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Mixed duopolies with advance production

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Abstract

Production to order and production in advance have been compared in many frameworks. In this paper we investigate a production in advance version of the capacity-constrained Bertrand-Edgeworth mixed duopoly game and determine the solution of the respective timing game. We show that a pure-strategy (subgame-perfect) Nash-equilibrium exists for all possible orderings of moves. It is pointed out that unlike the production-to-order case, the equilibrium of the timing game lies at simultaneous moves. An analysis of the public firm's impact on social surplus is also carried out. All the results are compared with those of the production-to order version of the respective game and with those of the mixed duopoly timing games.

Keywords: Bertrand-Edgeworth, mixed duopoly, timing games.
JEL Classification Number: D43, L13.

1 Introduction

We can distinguish between production-in-advance (PIA) and production-to-order (PTO) concerning how the firms organize their production in order to satisfy the consumers' demand.[1] In the former case production takes place before sales are realized, while in the latter one sales are determined before production takes place. Markets of perishable goods are usually mentioned as examples of advance production in a market. Phillips, Menkhaus, and Krogmeier (2001) emphasized that there are also goods which can be traded both in a PIA and in a PTO environment since PIA markets can be regarded as a kind of spot market whereas PTO markets as a kind of forward market. For example, coal and electricity are sold in both types of environments.

The comparison of the PIA and PTO environments has been carried out in experimental and theoretical frameworks for standard oligopolies.[2] For instance, assuming strictly increasing marginal cost functions Mestelman, Welland, and Welland (1987) found that in

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[1]The PIA game is also frequently called the price-quantity game or briefly PQ-game.

[2]We call an oligopoly standard if all firms are profitmaximizers, which basically means that they are privately owned.

an experimental posted offer market the firms' profits are lower in case of PIA. For more recent experimental analyses of the PIA environment we refer to Davis (2013) and Orland and Selten (2016). In a theoretical paper Shubik (1955) investigated the pure-strategy equilibrium of the PIA game and conjectured that the profits will be lower in case of PIA than in case of PTO. Levitan and Shubik (1978) and Gertner (1986) determined the mixed-strategy equilibrium for the constant unit cost case without capacity constraints.[3] Assuming constant unit costs and identical capacity constraints, Tasnádi (2004) found that profits are identical in the two environments and that prices are higher under PIA than under PTO. Zhu, Wu, and Sun (2014) showed for the case of strictly convex cost functions that PIA equilibrium profits are higher than PTO equilibrium profits. In addition, considering different orders of moves and asymmetric cost functions Zhu, Wu, and Sun (2014) demonstrated that the leader-follower PIA game leads to higher profit than the simultaneous-move PIA game.[4]

Concerning our theoretical setting, the closest paper is Tasnádi (2004) since we will investigate the constant unit cost case with capacity constraints. The main difference is that we will replace one profit-maximizing firm with a social surplus maximizing firm, that is we will consider a so-called mixed duopoly. We have already considered the PTO mixed duopoly in Balogh and Tasnádi (2012) for which we found (i) the payoff equivalence of the games with exogenously given order of moves, (ii) an increase in social surplus compared with the standard version of the game, and (iii) that an equilibrium in pure strategies always exists in contrast to the standard version of the game.[5] In this paper we demonstrate for the PIA mixed duopoly the existence of an equilibrium in pure strategies, (weakly) lower social surplus than in case of the PTO mixed duopoly and the emergence of simultaneous moves as a solution of a timing game.

It is also worthwhile to relate our paper briefly to the literature on mixed oligopolies. In a seminal paper Pal (1998) investigates for mixed oligopolies the endogenous emergence of certain orders of moves. Assuming linear demand and constant marginal costs, he shows for a quantity-setting oligopoly with one public firm that, in contrast to our result, the simultaneous-move case does not emerge. Matsumura (2003) relaxes the assumptions of linear demand and identical marginal costs employed by Pal (1998). The case of increasing marginal costs in Pal's (1998) framework has been investigated by Tomaru and Kiyono (2010). In line with our result on the timing of moves Bárcena-Ruiz (2007) obtained the endogenous emergence of simultaneous moves for a heterogeneous goods price-setting mixed duopoly timing game. In case of emission taxes Lee and Xu (2018) find that the sequential-move (simultaneous-move) game emerges in the equilibrium of the mixed duopoly timing game under significant (insignificant) environmental externality. There is also an evolving literature on managerial mixed duopolies, for instance, Nakamura (2018) shows that in this case a sequential order of moves emerges in which the private firm with a price contract moves first, while the public firm with a quantity contract moves second.

[3]Gertner (1986) also derived some important properties of the mixed-strategy equilibrium of the PIA game for strictly convex cost functions. For more on the PIA case see also Bos and Vermeulen (2015), van den Berg and Bos (2017), and Montez and Schutz (2018).

[4]From the mentioned papers only Zhu, Wu, and Sun (2014) considered sequential orders of moves. For more on standard duopoly leader-follower games we refer to Boyer and Moreaux (1987), Deneckere and Kovenock (1992) and Tasnádi (2003) in the Bertrand-Edgeworth framework. Furthermore, Din and Sun (2016) extended Zhu, Wu, and Sun (2014) to mixed duopolies.

[5]We refer the reader also to Bakó and Tasnádi (2017) which proves the validity of the Kreps and Scheinkman (1983) result for mixed duopolies by employing the Kreps and Scheinkman tie-breaking rule at the price-setting stage.

The remainder of the paper is organized as follows. In Section 2 we present our framework, Sections 3-5 contain the analysis of the three games with exogenously given order of moves, Section 6 solves the timing game, and we conclude in Section 7.

2 The framework

The demand is given by function D on which we impose the following restrictions:

Assumption 1. The demand function D intersects the horizontal axis at quantity a and the vertical axis at price b. D is strictly decreasing, concave and twice continuously differentiable on $(0, b)$; moreover, D is right-continuous at 0, left-continuous at b and $D(p) = 0$ for all $p \geq b$.

Clearly, any price-setting firm will not set its price above b. Let us denote by P the inverse demand function. Thus, $P(q) = D^{-1}(q)$ for $0 < q \leq a$, $P(0) = b$, and $P(q) = 0$ for $q > a$.

On the producers side we have a public firm and a private firm, that is, we consider a so-called mixed duopoly. We label the public firm as 1 and the private firm as 2. Henceforth, we will also label the two firms as i and j, where $i, j \in \{1, 2\}$ and $i \neq j$. Our assumptions imposed on the firms' cost functions are as follows:

Assumption 2. The two firms have identical $c \in (0, b)$ unit costs up to the positive capacity constraints k_1, k_2 respectively.

We shall denote by p^c the market clearing price and by p^M the price set by a monopolist without capacity constraints, i.e. $p^c = P(k_1 + k_2)$ and $p^M = \arg\max_{p\in[0,b]}(p - c)D(p)$. In what follows $p_1, p_2 \in [0, b]$ and $q_1, q_2 \in [0, a]$ stand for the prices and quantities set by the firms.

For any firm i and for any quantity q_j set by its opponent j we shall denote by $p_i^m(q_j)$ the profit maximizing price on firm i's residual demand curve $D_i^r(p, q_j) = (D(p) - q_j)^+$ with respect to its capacity constraint, i.e. $p_i^m(q_j) = \arg\max_{p\in[0,b]}(p - c)\min\{D_i^r(p, q_j), k_i\}$. Clearly, p_i^m is well defined whenever $c < P(q_j)$ and Assumptions 1-2 are satisfied. If $c \geq P(q_j)$, then $p_i^m(q_j)$ is not unique, as any price $p_i \in [0, b]$ together with quantity $q_i = 0$ results in $\pi_i = 0$ and π_i cannot be positive. For notational convenience we define $p_i^m(q_j)$ by b in case of $c \geq P(q_j)$.

For a given quantity q_j we shall denote the inverse residual demand curve of firm i by $R_i(\cdot, q_j)$. In addition, we shall denote by $q_i^m(q_j)$ the profit maximizing quantity on firm i's inverse residual demand curve subject to its capacity constraint, i.e. $q_i^m(q_j) = \arg\max_{q\in[0,k_i]}(R_i(q, q_j) - c)q$. It can be checked that $R_i(q_i, q_j) = P(q_i + q_j)$ and $q_i^m(q_j) = D_i^r(p_i^m(q_j), q_j)$.[6]

Let us denote by $p_i^d(q_j)$ the smallest price for which

$$(p_i^d(q_j) - c)\min\left\{k_i, D\left(p_i^d(q_j)\right)\right\} = (p_i^m(q_j) - c)q_i^m(q_j),$$

whenever this equation has a solution.[7] Provided that the private firm has 'sufficient' capacity, that is $\max\{p^c, c\} < p_2^m(k_1)$, then if it is a profit-maximizer, it is indifferent to

[6]Note that $D_i^r(p_i^m(q_j), q_j) \leq k_i$ since $p_i^m(q_j) \geq P(k_i + q_j)$.

[7]The equation defining $p_i^d(q_j)$ has a solution for any $q_j \in [0, k_j]$ if, for instance, $p_i^m(q_j) \geq \max\{p^c, c\}$, which will be the case in our analysis when we refer to $p_i^d(q_j)$.

whether serving residual demand at price level $p_2^m(q_1)$ or selling $\min\{k_2, D\left(p_2^d(q_1)\right)\}$ at the weakly lower price level $p_2^d(q_1)$. Observe that if $R_i(k_i, q_j) = p_i^m(q_j)$, then $p_i^d(q_j) = p_i^m(q_j)$.[8] We shall denote by $\tilde{q}_j$ the largest quantity for which $q_i^m(\tilde{q}_j) = k_i$ in case of $p^M \leq P(k_i)$ (i.e. $q_i^m(0) = k_i$), and zero otherwise. From Deneckere and Kovenock (1992, Lemma 1) it follows that $p_i^d(\cdot)$ and $p_i^m(\cdot)$ are strictly decreasing on $[\tilde{q}_j, k_j]$. Moreover, $q_i^m(\cdot)$ is strictly decreasing on $[\tilde{q}_j, k_j]$ and constant on $[0, \tilde{q}_j]$, and therefore $\tilde{q}_j = \inf\{q_j \in [0, a] \mid q_i^m(q_j) < k_i\}$ is always uniquely defined.

We assume efficient rationing on the market, and thus, the firms' demands equal

$$\Delta_i\left(D, p_1, q_1, p_2, q_2\right) = \begin{cases} D\left(p_i\right) & \text{if} \quad p_i < p_j, \\ T_i(p, q_1, q_2), & \text{if} \quad p = p_i = p_j \\ \left(D\left(p_i\right) - q_j\right)^+ & \text{if} \quad p_i > p_j, \end{cases}$$

for all $i \in \{1, 2\}$, where T_i stands for a tie-breaking rule. We will consider two sequential-move games (one with the public firm as the first mover and one with the private firm as the first-mover) and a simultaneous-move game. We employ the same tie-breaking rule as Deneckere and Kovenock (1992).

Assumption 3. If the two firms set the same price, then we assume for the sequential-move games that the demand is allocated first to the second mover[9] and for the simultaneous-move game that the demand is allocated in proportion of the firms' capacities.

Now we specify the firms' objective functions. The public firm aims at maximizing total surplus, that is,

$$\begin{aligned} \pi_1(p_1, q_1, p_2, q_2) &= \int_0^{\min\left\{(D(p_j)-q_i)^+, q_j\right\}} R_j(q, q_i)dq + \int_0^{\min\{a, q_i\}} P(q)dq - c(q_1 + q_2) \\ &= \begin{cases} \int_0^{\min\{D(p_j), q_1+q_2\}} P(q)dq - c(q_1 + q_2) & \text{if } D(p_j) > q_i, \\ \int_0^{\min\{D(p_i), q_i\}} P(q)dq - c(q_1 + q_2) & \text{if } D(p_j) \leq q_i, \end{cases} \end{aligned} \tag{1}$$

where $0 \leq p_i \leq p_j \leq b$. We illustrate social surplus in Figure 1.

The private firm is a profitmaximizer, and therefore,

$$\pi_2(p_1, q_1, p_2, q_2) = p_2 \min\left\{q_2, \Delta_2\left(D, p_1, q_1, p_2, q_2\right)\right\} - cq_2. \tag{2}$$

We divide our analysis into three cases.

1. The *strong private firm case*, where we assume that $q_2^m(k_1) < k_2$ and $P(k_1) > c$. This means that the private firm's capacity is large enough to have strategic influence on the outcome and the public firm cannot capture the entire market.

2. The *weak private firm case*, where we assume that $q_2^m(k_1) = k_2$ and $P(k_1) > c$. In this case the private firm's capacity is not large enough to have strategic influence on the outcome, but it has a unique profit-maximizing price on the residual demand curve.

[8]This can be the case if $p^M < P(k_1)$.

[9]This ensures for the case when the public firm moves first the existence of a subgame perfect Nash equilibrium in order to avoid the consideration of ε-equilibria implying a more difficult analysis without substantial gain.

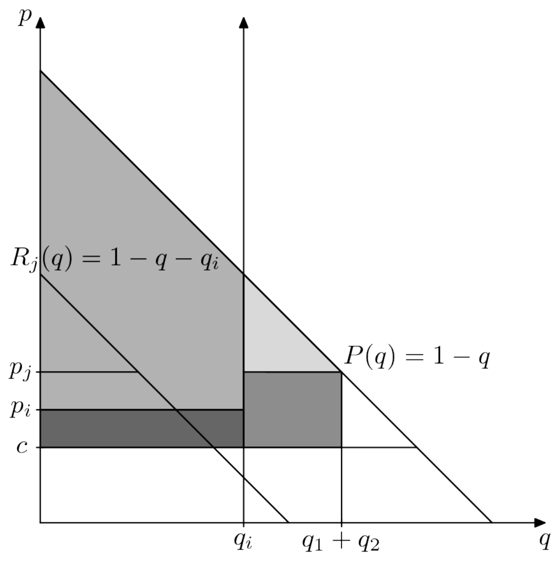

Figure 1: Social surplus

3. The *high unit cost case*, where we assume that $c \geq P(k_1)$. In this case if the public firm produces at its capacity level, then there is no incentive for the private firm to enter the market, because the cost level is too high.

Clearly, the three cases are well defined and disjunct from each other.

We now determine all the equilibrium strategies of both firms for the three possible orderings of moves in each of the three main cases. Within every case we begin with the simultaneous moves subcase, thereafter we focus on the public-firm-moves-first subcase, finally we analyze the private-firm-moves-first subcase. The results are always illustrated with numerical examples. For better visibility, the most interesting equilibria are depicted.

3 The strong private firm case

The following two inequalities remain true for the simultaneous moves and public leadership cases.

Lemma 1. *Under Assumptions 1-3,* $q_2^m(k_1) < k_2$ *and* $P(k_1) > c$ *we must have in case of simultaneous moves and public leadership that*

$$p_2^* \geq p_2^d(q_1^*) \tag{3}$$

in any equilibrium $(p_1^*, q_1^*, p_2^*, q_2^*)$ *in which* $q_1^* > 0$.

Proof. We obtain the result directly from the definition of $p_2^d(q_1)$. For any $q_1 \in [0, k_1]$, the private firm is better off by setting $p_2 = p_2^m(q_1)$ and $q_2 = q_2^m(q_1)$ than by setting any price $p_2 < p_2^d(q_1)$ and any quantity $q_2 \in [0, k_2]$. □

Lemma 2. *Under Assumptions 1-3,* $q_2^m(k_1) < k_2$ *and* $P(k_1) > c$ *we have in case of simultaneous moves and public leadership that*

$$p_2^* \leq p_2^m(0) = \max\{P(k_2), p^M\} \tag{4}$$

in any equilibrium $(p_1^*, q_1^*, p_2^*, q_2^*)$.

Proof. Suppose that $p_2^* > \max\{P(k_2), p^M\}$. If $p_2^* \leq p_1^*$, then the private firm would be better off by setting price $\max\{P(k_2), p^M\}$ and quantity $D\left(\max\{P(k_2), p^M\}\right)$. If $p_2^* > p_1^*$, then the private firm serves residual demand, and therefore it could benefit from switching to action $(p_2^m(q_1^*), q_2^m(q_1^*))$, $\left(\max\{P(k_2), p^M\}, D\left(\max\{P(k_2), p^M\}\right)\right)$, or $(p_1^* - \varepsilon, \min\{k_2, D(p_1^* - \varepsilon)\})$, where ε is a sufficiently small positive value. For both cases we have obtained a contradiction. □

3.1 Simultaneous moves

For the case of simultaneous moves we have a pure-strategy Nash equilibrium family,[10] which contains profiles where the private firm maximizes its profit on the residual demand choosing $p_2^* = p_2^m(q_1^*)$ and $q_2^* = q_2^m(q_1^*)$, while the public firm can choose any price level not greater than $p_2^d(q_1^*)$ and produce any non-negative amount up to its capacity. It is worth emphasizing that in case of $p_2^m(q_1^*) = p_2^d(q_1^*)$ the private firm can sell its entire capacity.

Proposition 1 (Simultaneous moves)**.** *Let Assumptions 1-3, $q_2^m(k_1) < k_2$ and $P(k_1) > c$ be satisfied. A strategy profile*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1^*, q_1^*, p_2^m(q_1^*), q_2^m(q_1^*)) \tag{5}$$

is for a quantity $q_1^* \in (0, k_1]$ and for any price $p_1^* \in \left[0, p_2^d(q_1^*)\right]$ or for any $q_1^* = 0$ and any $p_1^* \in [0, b]$ a Nash-equilibrium in pure strategies if and only if

$$\pi_1\left(p_2^d(q_1^*), q_1^*, p_2^m(q_1^*), q_2^m(q_1^*)\right) \geq \pi_1(P(k_1), k_1, p_2^m(q_1^*), q_2^m(q_1^*)), \text{[11]} \tag{6}$$

where there exists a nonempty closed subset H of $[0, k_1]$ satisfying condition (6).[12] *Finally, no other equilibrium in pure strategies exists.*

Proof. Assume that $(p_1^*, q_1^*, p_2^*, q_2^*)$ is an arbitrary equilibrium profile. We divide our analysis into three subcases. In the first case (Case A) we have $p_1^* = p_2^*$, in the second one (Case B) $p_1^* > p_2^*$ holds true, while in the remaining case we have $p_1^* < p_2^*$ (Case C).

Case A: We claim that $p_1^* = p_2^*$ implies $q_1^* + q_2^* = D(p_2^*)$. Suppose that $q_1^* + q_2^* < D(p_2^*)$. Then[13] because of $p_2^* > \max\{p^c, c\}$ by a unilateral increase in output the public firm could increase social surplus or the private firm could increase its profit; a contradiction. Suppose that $q_1^* + q_2^* > D(p_2^*)$. Then the public firm could increase social surplus by decreasing its output or if $q_1^* = 0$, the private firm could increase its profit by producing only $D(p_2^*)$; a contradiction.

We know that we must have $p_1^* = p_2^* \geq p_2^d(q_1^*)$ by Lemma 1. Assume that $q_1^* > 0$. Then we must have $q_2^* = \min\{k_2, D(p_2^*)\}$, since otherwise the private firm could benefit from reducing its price slightly and increasing its output sufficiently (in particular, by setting $p_2 = p_2^* - \varepsilon$ and $q_2^* = \min\{k_2, D(p_2)\}$). Observe that $p_2^m(0) = p_2^d(0)$, $p_2^m(q_1) = p_2^d(q_1)$ for all $q_1 \in [0, \tilde{q}_1]$ and $p_2^m(q_1) > p_2^d(q_1)$ for all $q_1 \in (\tilde{q}_1, k_1]$.[14] Moreover, it can be verified by the definitions of $p_2^m(q_1^*)$ and $p_2^d(q_1^*)$ that $q_1^* + k_2 \geq D(p_2^d(q_1^*)) \geq D(p_2^*)$, where the first inequality is strict if $q_1^* > \tilde{q}_1$. Thus, $q_1^* > \tilde{q}_1$ is in contradiction with $q_2^* = \min\{k_2, D(p_2^*)\}$

[10] Provided that certain conditions hold true.

[11] Clearly, $P(k_1) < p_2^m(q_1^*)$, i.e. $k_1 > D(p_2^m(q_1^*)) = q_2^m(q_1^*)$ is a necessary condition for (6).

[12] In particular, there exists a subset $[\overline{q}, k_1]$ of H.

[13] Observe that by Lemma 1, the monotonicity of $p_2^d(\cdot)$, $q_2^m(k_1) < k_2$ and $P(k_1) > c$, we have $p_2^* \geq p_2^d(q_1) \geq p_2^d(k_1) > \max\{p^c, c\}$.

[14] We recall that $\tilde{q}_i$ has been defined after $p_i^d(q_j)$.

since we already know that $q_1^* + q_2^* = D(p_2^*)$ in Case A. Hence, an equilibrium in which both firms set the same price and the public firm's output is positive exists if and only if $p_2^m(q_1^*) = p_2^d(q_1^*)$ (i.e., $q_1^* \in (0, \tilde{q}_1)$) and (6) is satisfied. This type of equilibrium appears in (5) with $q_2^* = q_2^m(q_1^*) = k_2$.

Moreover, it can be verified that $(p_1^*, q_1^*, p_2^*, q_2^*) = (p_2^m(0), 0, p_2^m(0), q_2^m(0))$ is an equilibrium profile in pure strategies if and only if

$$\pi_1(p_2^m(0), 0, p_2^m(0), q_2^m(0)) \geq \pi_1(P(k_1), k_1, p_2^m(0), q_2^m(0)), \tag{7}$$

where we emphasize that $p_2^m(0) = \max\{P(k_2), p^M\}$ and $q_2^m(0) = D(\max\{P(k_2), p^M\})$.

Case B: Suppose that $p_1^* > p_2^* \geq p_2^d(q_1^*)$ and $D(p_2^*) > q_2^*$. Then the public firm could increase social surplus by setting price $p_1 = p_2^*$ and $q_1 = \min\{k_1, D(p_2^*) - q_2^*\}$; a contradiction.

Assume that $p_1^* > p_2^* \geq p_2^d(q_1^*)$ and $D(p_2^*) = q_2^*$. Then in an equilibrium we must have $q_1^* = 0$, $p_2^* = p_2^m(0)$ and $q_2^* = q_2^m(0)$. Furthermore, it can be checked that these profiles specify equilibrium profiles if and only if equation (6) is satisfied.

Clearly, $p_1^* > p_2^* \geq p_2^d(q_1^*)$ and $D(p_2^*) < q_2^*$ cannot be the case in an equilibrium since the private firm could increase its profit by producing $q_2 = D(p_2^*)$ at price p_2^*. Finally, by Lemma 1 $p_2^* < p_2^d(q_1^*)$ cannot be the case either.

Case C: In this case $p_2^* = p_2^m(q_1^*)$ and $q_2^* = q_2^m(q_1^*)$ must hold, since otherwise the private firm's payoff would be strictly lower. In particular, if the private firm sets a price not greater than p_1^*, we are not anymore in Case C; if $q_2^* > \min\{D_2^r(p_2^*, q_1^*), k_2\}$, then the private firm either produces a superfluous amount or is capacity constrained; if $q_2^* < \min\{D_2^r(p_2^*, q_1^*), k_2\}$, then the private firm could still sell more than q_2^*; and if $q_2^* = \min\{D_2^r(p_2^*, q_1^*), k_2\}$, then the private firm will choose a price-quantity pair maximizing profits with respect to its residual demand curve $D_2^r(\cdot, q_1^*)$ subject to its capacity constraint. In addition, in order to prevent the private firm from undercutting the public firm's price we must have $p_1^* \leq p_2^d(q_1^*)$.

Clearly, for the given values p_1^*, p_2^* and q_2^* from our equilibrium profile the public firm has to choose a quantity $q_1' \in [0, k_1]$, which maximizes function $f(q_1) = \pi_1(p_1^*, q_1, p_2^*, q_2^*)$ on $[0, k_1]$. We show that $q_1' = q_1^*$ must be the case. Obviously, it does not make sense for the public firm to produce less than q_1^* since this would result in unsatisfied consumers. Observe that for all $q_1 \in [q_1^*, \min\{D(p_2^*), k_1\}]$

$$\begin{aligned} f(q_1) &= \int_0^{D(p_2^*)-q_1} (R_2(q, q_1) - c)\, dq + \int_0^{q_1} (P(q) - c)\, dq - c(q_1 - q_1^*) = \\ &= \int_0^{D(p_2^*)} P(q) dq - D(p_2^*) c - c(q_1 - q_1^*). \end{aligned} \tag{8}$$

Since only $-c(q_1 - q_1^*)$ is a function of q_1 we see that f is strictly decreasing on $[q_1^*, \min\{D(p_2^*), k_1\}]$.

Subase (i): In case of $k_1 \leq D(p_2^*)$ we have already established that q_1^* maximizes f on $[0, k_1]$. Moreover, (p_1^*, q_1^*) maximizes $\pi_1(p_1, q_1, p_2^*, q_2^*)$ on $[0, p_2^*) \times [0, k_1]$ since equation (8) is not a function of p_1^*. Hence, for any $p_1 \leq p_2^d(q_1^*)$ such that $p_1 < p_2^*$ we have that $(p_1, q_1^*, p_2^m(q_1^*), q_2^m(q_1^*))$ specifies a Nash equilibrium for any $q_1 \in [0, k_1]$ satisfying $k_1 \leq D(p_2^m(q_1^*))$. However, note that in case of $q_1^* \in [0, \tilde{q}_1]$ and $p_1 = p_2^d(q_1^*)$ we are leaving Case C and obtain a Case A Nash equilibrium.

Observe that $p_2^m(k_1) > \max\{p^c, c\}$ implies that $k_1 < D(p_2^m(k_1))$, and therefore we always have Subcase (i) equilibrium profiles. Since $D(p_2^m(\cdot))$ is a continuous and strictly increasing function, interval $[\widetilde{q}_1, k_1] \cap (0, k_1]$ determines the set of quantities yielding an equilibrium for Subcase (i).

Subase (ii): Turning to the more complicated case of $k_1 > D(p_2^*)$, we also have to investigate function f above the interval $[D(p_2^*), k_1]$ in which region the private firm does not sell anything at all at price p_2^* and

$$f(q_1) = \int_0^{\min\{q_1, D(p_1^*)\}} (P(q) - c)\, dq - cq_2^* - c\,(q_1 - D(p_1^*))^+ . \tag{9}$$

Observe that we must have $P(k_1) < p_2^*$. If the public firm is already producing quantity $q_1 = D(p_2^*)$, the private firm does not sell anything at all and contributes to a social loss of cq_2^*. Therefore, $f(q)$ is increasing on $[D(p_2^*), \min\{D(p_1^*), k_1\}]$.

Assume that $k_1 \leq D(p_1^*)$. Then for any $p_1 \leq p_2^d(q_1^*)$ we get that $(p_1, q_1^*, p_2^m(q_1^*), q_2^m(q_1^*))$ is a Nash equilibrium if and only if

$$\begin{aligned} \pi_1\left(p_2^d(q_1^*), q_1^*, p_2^m(q_1^*), q_2^m(q_1^*)\right) &\geq \pi_1\left(p_2^d(q_1^*), k_1, p_2^m(q_1^*), q_2^m(q_1^*)\right) = \\ &= \pi_1\left(P(k_1), k_1, p_2^m(q_1^*), q_2^m(q_1^*)\right), \end{aligned} \tag{10}$$

where the last equality follows from the inequalities $p_1^* \leq P(k_1) \leq p_2^*$ valid for this case and the fact that social surplus is maximized in function of (p_1, q_1) subject to the constraint that the private firm does not sell anything at all if the public firm sets an arbitrary price not greater than $P(k_1)$ and produces k_1.

Assume that $k_1 > D(p_1^*)$. Therefore, $f(q)$ would be decreasing on $[D(p_1^*), k_1]$. However, it can be checked that the public firm could increase social surplus by switching to strategy $(P(k_1), k_1)$ from strategy $(p_1^*, D(p_1^*))$. In addition, any strategy (p_1, k_1) with $p_1 \leq P(k_1)$ maximizes social surplus subject to the constraint that the private firm does not sell anything at all. Therefore, $\left(p_2^d(q_1^*), q_1^*, p_2^m(q_1^*), q_2^m(q_1^*)\right)$ is a Nash equilibrium if and only if condition (6) is satisfied. Comparing equation (10) with equation (6), we can observe that we have derived the same necessary and sufficient condition for a strategy profile being a Nash equilibrium, which is valid for Subcase (ii).

So far we have established that there exists a function g, which uniquely determines the highest equilibrium price as a function of quantity q produced by the public firm. Clearly, $g(q) = p_2^d(q)$, where the domain of g is not entirely specified. At least we know from Subcase (i) that the domain of g contains $[\widetilde{q}_1, k_1]$. Observe also that the equilibrium profiles of Subcase (i) satisfy condition (6). Let $u(q_1) = \pi_1\left(p_2^d(q_1), q_1, p_2^m(q_1), q_2^m(q_1)\right)$ and $v(q_1) = \pi_1(P(k_1), k_1, p_2^m(q_1), q_2^m(q_1))$. Hence, q_1 determines a Nash equilibrium profile if and only if $u(q_1) \geq v(q_1)$. It can be verified that u and v are continuous, and therefore, set $H = \{q \in [0, k_1] \mid u(q) \geq v(q)\}$ is a closed set containing $[\widetilde{q}, k_1]$. □

For the illustration of the Nash equilibrium profile mentioned in the statement let $D(p) = 1 - p$, $k_1 = 0.5$, $k_2 = 0.4$, and $c = 0.1$. Now the following values can be calculated directly from the exogenously given data: $p^c = 0.1$, $p_2^m(k_1) = 0.3$, $q_2^m(k_1) = 0.2$, $p_2^d(k_1) = 0.2$. Since $p_2^m(k_1) > p_2^d(k_1)$ we have

$$(p_1^*, q_1^*, p_2^*, q_2^*) = \left(p_1^*, q_1^*, \frac{1 - q_1^* - c}{2}, \frac{1 - q_1 + c}{2}\right)$$

in equilibrium, where $q_1^* \in [0, 0.5]$ and $p_1^* \in [0, 0.2]$.

In particular, if $q_1^* = k_1 = 0.5$ and $p_1^* = p_2^d(k_1) = 0.2$, then $p_2^* = 0.3$ and $q_2^* = 0.2$ (see Figure 2). Calculating the social surplus (the sum of dark gray and light gray areas in Figure 2) and the private firm's profit (the light gray area indicated by π_2), we obtain $\pi_1 = 0.435$ and $\pi_2 = 0.04$. It is easy to check that for this profile the necessary condition (6) is satisfied.

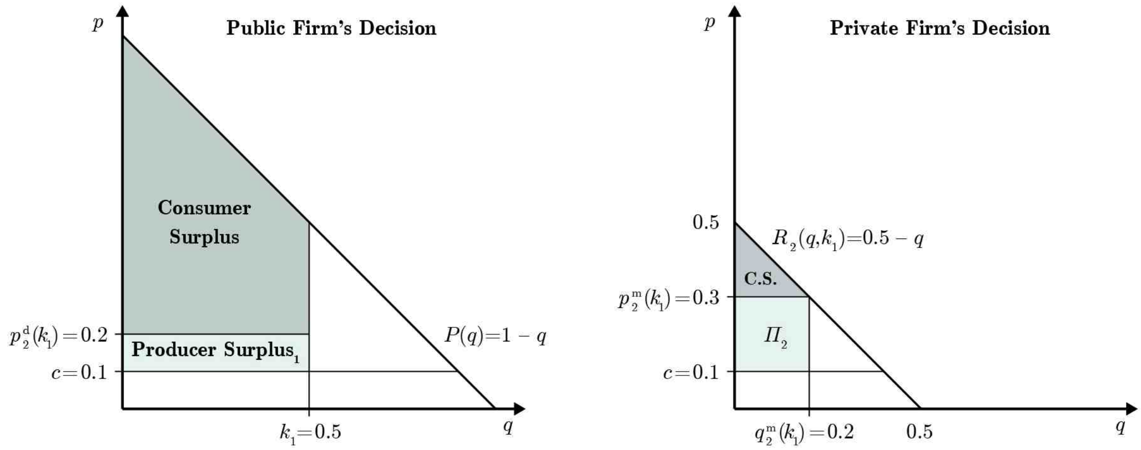

Figure 2: The strong private firm case - both firms have positive output

Clearly, p_1^* and q_1^* can vary within the given ranges. Decreasing p_1^* results in lower producer surplus for the public firm, but in an equally large increase in consumer surplus. Thus, payoffs remain the same. Altering q_1^* shifts the residual demand curve, and results in varying payoffs. The possible payoff intervals can also be calculated for the example: $\pi_1 \in [0.28, 0.435]$ and $\pi_2 \in [0.04, 0.2]$.

3.2 Public firm moves first

We continue with the case of public leadership. Here, we have a unique family of pure-strategy subgame-perfect Nash equilibria, where the public firm produces its capacity limit at a price not greater than $p_2^d(k_1)$. The private firm serves residual demand and acts as a monopolist on the residual demand curve, as presented in the following proposition.

Proposition 2 (Public firm moves first)**.** *Let Assumptions 1-3, $q_2^m(k_1) < k_2$ and $P(k_1) > c$ be satisfied. Then the set of SPNE prices and quantities are given by*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, k_1, p_2^m(k_1), q_2^m(k_1)) \tag{11}$$

for any $p_1 \leq p_2^d(k_1)$.

Proof. First, we determine the best reply $BR_2 = (p_2^*(\cdot,\cdot), q_2^*(\cdot,\cdot))$ of the private firm. Observe that the private firm's best response correspondence can be obtained from the proof of Proposition 1. $BR_2(p_1, q_1) =$

$$\begin{cases} \{(p_2^m(q_1), q_2^m(q_1))\} & \text{if } p_1 < p_2^d(q_1); \\ \{(p_2^m(q_1), q_2^m(q_1)), (p_1, \min\{k_2, D(p_1)\})\} & \text{if } p_1 = p_2^d(q_1); \\ \{(p_1, \min\{k_2, D(p_1)\})\} & \text{if } p_2^d(q_1) < p_1 \leq p_2^m(0); \\ \{(p_2^m(0), q_2^m(0))\} & \text{if } p_2^m(0) < p_1. \end{cases}$$

Though there are two possible best replies for the private firm to the public firm's first-period action $\left(p_2^d(q_1), q_1\right)$, in an SPNE the private firm must respond with $(p_2^m(q_1), q_2^m(q_1))$ because otherwise, there will not be an optimal first-period action for the public firm. Hence, the public firm maximizes social surplus in the first period by choosing price $p_1^* = p_2^d(k_1)$ and quantity k_1. Then the private firm follows with price $p_2^* = p_2^m(k_1)$ and quantity $q_2^* = q_2^m(k_1)$. □

Continuing with the example of linear demand $D(p) = 1 - p$, we focus on the simultaneous-move outcome, which matches the SPNE emerging in case of public leadership. Let the capacities and the unit cost be $k_1 = 0.5$, $k_2 = 0.4$ and $c = 0.1$, respectively.

Then the actions associated with the only subgame-perfect Nash equilibrium profile are

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1^*, 0.5, 0.3, 0.2)\,.$$

where $p_1^* \in [0, 0.3]$. The social surplus and the private firm's profit are equal to $\pi_1 = 0.435$ and $\pi_2 = 0.04$.

3.3 Private firm moves first

Now we consider the case of private leadership. In this case, there exists one type of subgame-perfect Nash equilibria in which the private firm produces on the original demand curve at the highest price level not above its its monopoly price for which it is still of the public firm's interest to remain on the residual demand curve and produce less than it would produce on the original demand curve. Formally, the private firm sets price

$$\begin{aligned}\widetilde{p}_2 = \max\{p_2 \in [c, p_2^m(0)] \quad | \quad & \pi_1(p_1, D_1^r(p_2, \min\{D(p_2), k_2\}), p_2, \min\{D(p_2), k_2\}) \geq \\ & \pi_1(P(k_1), k_1, p_2, \min\{D(p_2), k_2\})\}\end{aligned}$$

in the first stage. The equilibrium profiles with their necessary conditions are given formally in the following proposition and the existence of the price $\widetilde{p}_2$ is shown in its proof.

Proposition 3 (Private firm moves first)**.** *Let Assumptions 1-3, $q_2^m(k_1) < k_2$ and $P(k_1) > c$ be satisfied. The equilibrium actions of the firms associated with an SPNE are the following ones*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, D_1^r(\widetilde{p}_2, \min\{D(\widetilde{p}_2), k_2\}), \widetilde{p}_2, \min\{D(\widetilde{p}_2), k_2\}) \tag{12}$$

where $p_1^* \in [0, \widetilde{p}_2]$ can be an arbitrary price; furthermore, $p_1^* \in (\widetilde{p}_2, b]$ are also equilibrium prices in case of $q_1^* = 0$.

Proof. We determine the SPNE of the private leadership game by backwards induction without explicitly referring to the proof of Proposition 1. For any given first-stage action (p_2, q_2) of the private firm the public firm never produces less than $\min\{D_1^r(p_2, q_2), k_1\}$ in the second stage. Moreover, if the public firm does not capture the entire market (i.e. the private firm's sales are positive), it never produces more than $\min\{D_1^r(p_2, q_2), k_1\}$. If

$$\pi_1(p_2, \min\{D_1^r(p_2, q_2), k_1\}, p_2, q_2) \geq \pi_1(P(k_1), k_1, p_2, q_2) \tag{13}$$

is satisfied at a price $p_2 \in [0, b]$ and a quantity $q_2 \in (0, k_2]$, then the private firm, by choosing its first-stage action (p_2, q_2), becomes a monopolist on the market (in case of $q_2 \geq D(p_2)$) or sells its entire production (in case of $q_2 < D(p_2)$) since the public firm cannot increase social surplus by setting a lower price than p_2 and it definitely does not set a price above p_2. To be more precise if (13) is satisfied with equality the public firm could also respond with price $P(k_1)$ and quantity k_1; however, as it can be verified later in an SPNE the public firm does not choose the latter response. Clearly, if (13) is violated, the public firm responds with price $P(k_1)$ and quantity k_1. Therefore, we get $BR_1(p_2, q_2) =$

$$\begin{cases} \{(p_1, D_1^r(p_2, q_2) \mid p_1 \leq p_2)\} & \text{if } \pi_1(p_1, D_1^r(p_2, q_2), p_2, q_2) > \pi_1(P(k_1), k_1, p_2, q_2); \\ \{(p_1, k_1) \mid p_1 \leq P(k_1))\} & \text{if } \pi_1(p_1, D_1^r(p_2, q_2), p_2, q_2) < \pi_1(P(k_1), k_1, p_2, q_2); \\ \{(p_1, D_1^r(p_2, q_2) \mid p_1 \leq p_2)\} \cup & \\ \{(p_1, k_1) \mid p_1 \leq P(k_1))\} & \text{if } \pi_1(p_1, D_1^r(p_2, q_2), p_2, q_2) = \pi_1(P(k_1), k_1, p_2, q_2); \end{cases}$$

Clearly, the private firm does not set a price below c jointly with a positive quantity. Furthermore, the private firm can make positive profits because of $q_2^m(k_1) < k_2$ and $P(k_1) > c$, and therefore it sets a price above c. For any given $p_2 > c$ the private firm will never produce less than $\min\{D_2^r(p_2, k_1), k_2\}$ and the left hand side of (13) is constant in q_2 on $[\min\{D_2^r(p_2, k_1), k_2\}, \min\{D(p_2), k_2\}]$, while the profits of the private firm are strictly increasing in q_2 on the latter interval. Therefore, the private firm produces $q_2 = \min\{D(p_2), k_2\}$ if it produces at all. Henceforth, we substitute $q_2 = \min\{D(p_2), k_2\}$ in equation (13). Then the private firm would like to set price $p_2^m(0)$ if (13) is satisfied at this price level, otherwise it sets the highest price still satisfying (13). Note that (13) is definitely satisfied at price $p_2^m(0)$ if $P(k_1) \geq p_2^m(0)$, and otherwise the LHS of (13) is larger than its RHS at price $P(k_1)$, the LHS is strictly decreasing and continuous, while the RHS is strictly increasing and continuous on $[P(k_1), p_2^m(0)]$, and therefore if (13) is not satisfied at $p_2^m(0)$, there exists a unique price $\widetilde{p} \in [P(k_1), p_2^m(0))$ such that (13) is satisfied with equality at price $\widetilde{p}$. In the former case the private firm sets price $p_2^m(0)$, while in the latter case price $\widetilde{p}$ in the SPNE. □

To illustrate Proposition 3 take again the linear demand curve $D(p) = 1 - p$. First, let $k_1 = 0.5$, $k_2 = 0.4$ and $c = 0.1$ for which $\widetilde{p}_2 = p_2^m(0)$ will be the case. The following values can be calculated directly from the exogenously given data: $p^c = 0.1$, $p^M = 0.55$, $q^M = 0.45$, $P(k_2) = 0.6$. In what follows, in the first stage the private firm will set $p_2^* = P(k_2) = 0.6$ and $q_2^* = k_2 = 0.4$. It can be checked that for these values the public firm has no incentive to enter the market at stage two. Thus, the actions associated with the SPNE in this case are for all $p_1 \in [0, 1]$:

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0, 0.6, 0.4)$$

The respective payoffs are as follows: $\pi_1 = 0.28$ and $\pi_2 = 0.2$.

Second, let $k_1 = 0.5$, $k_2 = 0.4$ and $c = 0.1$ for which $\widetilde{p}_2 < p_2^m(0)$ will be the case. Then it can be checked that the public firm will enter the market. Being aware of this, the private firm sets the highest price level ($\widetilde{p}_2$) at which it can still sell its entire capacity so that the public firm has no incentive to undercut the price level set by the private firm. In this case $\widetilde{p}_2 = 0.487$. The public firm will then satisfy residual demand at $\widetilde{p}_2$ price level, i.e. $q_1^* = 0.213$. The public firm can set its price to any level within $[0, 0.487]$. To sum up, the actions associated with the SPNE in this case are as follows:

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0.213, 0.487, 0.4)\,,$$

where $p_1 \in 0, 0.487$. The payoffs are $\pi_1 = 0.36$ and $\pi_2 = 0.116$.

4 The weak private firm case

The main assumption throughout this section is that the private firm does not have sufficient capacity to influence the market strategically, that is why we call the private firm weak. Formally, $q_2^m(k_1) = k_2$, and in addition $P(k_1) > c$. We begin the analysis with the following lemma which dictates that the private firm is not intended to set any price below the market clearing price.

Lemma 3. *Assume that Assumptions 1-3, $q_2^m(k_1) = k_2$ and $P(k_1) > c$ hold true. Given any strategy (p_1, q_1) of the public firm, the private firm's strategies (p_2, q_2) with price level $p_2 < \max\{p^c, c\}$ and any quantity $q_2 > 0$ are strictly dominated, for instance by a strategy with $p_2 = \max\{p^c, c\}$ and $q_2 > 0$, in all three possible orderings.*

Proof. If $p_2 < \max\{p^c, c\}$, then the private firm can sell its entire capacity or makes losses, independently from the public firm's strategy. Clearly, given any (p_1, q_1) and $q_2 > 0$, replacing the private firm's price level by $p_2 = \max\{p^c, c\}$, π_2 increases, thus, the private firm's strategies with lower price levels become strictly dominated. □

4.1 Simultaneous moves

Here, we have two main types of subgame-perfect Nash equilibria. In the first type the private firm sets the highest price level at which it can still produce on the original demand curve. As a particular case of this equilibrium, clearing the market may emerge. The second type contains profiles for which the private firm is a monopolist on the original demand curve.

Proposition 4 (Simultaneous moves)**.** *Assume that Assumptions 1-3, $q_2^m(k_1) = k_2$ and $P(k_1) > c$ hold.A strategy profile*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1^*, D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\}), \hat{p}, \min\{k_2, D(\hat{p})\}) \tag{14}$$

where $p_1^* \in [0, \hat{p}]$ in case of $q_1^* > 0$ and $p_1^* \in [0, b]$ in case of $q_1^* = 0$, defines a Nash equilibrium family in pure strategies if and only if all of the following conditions hold:

$$p_2^m(0) \geq \hat{p} \geq p_2^m(q_1^*) \tag{15}$$

and

$$\pi_1(p^c, k_1, \hat{p}, q_2^*) \leq \pi_1(p_1^*, q_1^*, \hat{p}, q_2^*). \tag{16}$$

In particular, if $\hat{p} = p^c$, then $(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1^*, k_1, p^c, k_2)$ is a Nash equilibrium.

Proof. Assume that $(p_1^*, q_1^*, p_2^*, q_2^*)$ is an arbitrary equilibrium profile. It can be verified that $q_1^* + q_2^* = D(p')$, where p' stands for the highest price from p_1^*, p_2^* at which at least one firm sells a positive amount. Like in the analysis of the strong private firm case, we divide our analysis into three subcases. In the first case (Case A) we have $p_1^* = p_2^*$, in the second one (Case B) $p_1^* > p_2^*$ holds, while in the remaining case we have $p_1^* < p_2^*$ (Case C).

Case A: By Lemma 3 we have $p_1^* = p_2^* \geq p^c$. First, we verify that the strategy profile given by (14) is a Nash-equilibrium profile for any $\hat{p} \geq p^c$ if (15) and (16) are satisfied. Hence, firms set quantities $q_2^* = \min\{k_2, D(\hat{p})\}$ and $q_1^* = D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\})$. By the second inequality in (15), the private firm has no incentive to increase its price. If $D(\hat{p}) \geq k_2$, then decreasing p_2 is trivially irrational for the private firm that already sells its entire capacity. In case $k_2 > D(\hat{p})$, we obtain a particular equilibrium

$(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, 0, \hat{p}, D(\hat{p}))$, which means that the public firm is not present on the market, and therefore, by the first inequality in (15) the private firm has no incentive to decrease its price.

Now we consider the public firm's actions. Clearly, increasing the public firm's price would not increase, but in fact reduce total surplus if $q_1^* > 0$. Moreover, prices $p_1^* = p_2^* = p^c$ with quantities $q_1^* = D_1^r(\hat{p}, \min\{k_2, D(p^c)\}) = k_1$ and $q_2^* = \min\{k_2, D(p^c)\} = k_2$ would result in the best possible outcome for the public firm. Hence, we still have to investigate the effect of a potential price decrease by the public firm in case of $p_1^* = p_2^* > p^c$. If the public firm reduces its price without increasing its quantity, obviously total surplus cannot increase. To analyze the case in which the public firm decreases its price and increases its quantity at the same time, observe that the sum of consumer surplus and the two firms' revenues (which equals $\pi_1(p_1, q_1, p_2, q_2) + c(q_1 + q_2)$) is only a function of the highest price at which sales are still positive. Therefore, total surplus is strictly decreasing in q_1 on $(q_1^*, D(\hat{p}))$ and strictly increasing in q_1 on $[D(\hat{p}), k_1]$ for a given $p_1 < p_1^*$. To see the latter statement notice that within $[D(\hat{p}), k_1]$ the superfluous production of the private firm remains the same, that is its entire production. Hence, we have shown that the benchmark action of the public firm in order to determine whether the public firm has an incentive to reduce its price is (p^c, k_1), which is in line with (16).

Turning to the case where (15) is violated, we show that (14) cannot be a Nash-equilibrium profile. If $\hat{p} < p_2^m(q_1^*)$ the private firm will increase its price until $p_2^m(q_1)$ to become a monopolist on the residual demand curve, where we are not anymore in Case A of our analysis. Note that any $p_1^* \in [0, \hat{p}]$ results in the same outcome, but if $p_1^* \neq p_2^*$, we are again either in Case B or in Case C. If $p_2^m(0) < \hat{p}$, the private firm will switch to price $p_2^m(0)$.

As a special case of $\hat{p} = p^c$, clearing the market is always a Nash equilibrium for the following reason: by $p^c \geq p_2^m(k_1)$ the private firm cannot be better off by unilaterally increasing its price even by reducing its quantity, accordingly. Note that the market-clearing equilibrium ensures that an equilibrium in pure strategies always exists in the weak private firm case.

Now we show that no other equilibrium exists given that $p_1^* = p_2^* \geq p^c$. Assume that $q_2^* < \min\{k_2, D(p_1^*)\}$. In such cases the private firm gets better off by slightly undercutting p_1^* and selling $q_2^* = \min\{k_2, D(p_1^* - \varepsilon)\}$. Now assume that $q_1^* \neq D_1^r(p_1^*, \min\{k_2, D(p_1^*)\}$. If the left hand side is larger, then there is superfluous production that results in surplus loss; if the left hand side is smaller, then there is a loss in consumer surplus. Thus, there are no more equlibria, if $p_1^* = p_2^*$.

Case B: By Lemma 3 $p_1^* > p_2^* \geq p^c$. By decreasing p_1 to p_2^*, the public firm can always increase social surplus, unless $q_1^* = 0$. In the extreme case of $q_1^* = 0$, p_1^* can obviously be any nonnegative amount. Besides, if $k_2 \geq D(p_2^*)$ and (16) holds, we arrive to the Nash equilibria in which the private firm sets price $p_2^m(0)$. If $k_2 < D(p_2^*)$, then the public firm can increase social surplus by setting price $p_1 = p_2^*$ and quantity $q_1^* = D(p_2^*) - k_2$.

Case C: Now we have $p_2^* > p_1^*$. As already shown in Case A, this case emerges in equilibrium if $(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\}), \hat{p}, \min\{k_2, D(\hat{p})\})$, and $p_1^* < \hat{p}$, that is, we have the Nash equilibrium mentioned in the statement. It remains to show that there is no other possible equilibrium in this case. If $p_2^* > p_1^*$, then $p_2^* = p_2^m(q_1^*)$ and $q_2^* = D_2^r(p_2^*, q_1^*) = q_2^m(q_1^*)$ must hold, since otherwise the private firm's payoff would be strictly lower. The arguments for this are analogous to those mentioned in the strong private firm case.[15] As $q_2^m(k_1) = k_2$, due to the fact that $q_2^m(\cdot)$ is decreasing[16] in q_1, for any $q_1 < k_1$, $q_2^m(q_1) > q_2^m(k_1) = k_2$. Thus, q_2^* must equal k_2. It is easy to see that for this case the only possible type of equilibrium is characterized in the statement.

□

In case of linear demand $D(p) = 1 - p$ the weak private firm case emerges if $k_1 = 0.9$, $k_2 = 0.02$, $c = 0.01$. From these exogenously given values we can determine $p^c = 0.08$ and $\tilde{p_2} = 0.102$. In this case we have several Nash equilibrium profiles, which are not payoff equivalent. For all $\hat{p} \in [0.08, 0.102]$ and any $p_1 \in [0, \hat{p}]$,

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0.98 - \hat{p}, \hat{p}, 0.02)$$

defines the family of Nash equilibrium profiles. In particular, if $\hat{p} = p^c$, then

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0.9, 0.08, 0.02)$$

and the social surplus associated to the market clearing equilibrium is $\pi_1 = 0.4876$, while the private firm's profit is $\pi_2 = 0.0014$.

In the case in which the firms do not choose the market clearing price, let $\hat{p} = 0.102$ (see Figure 3). Then the equilibrium profile is

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0.878, 0.102, 0.02),$$

the corresponding payoffs are $\pi_1 = 0.4858$ (the sum of dark and light gray areas) and $\pi_2 = 0.0018$ (the light gray area indicated by π_2).

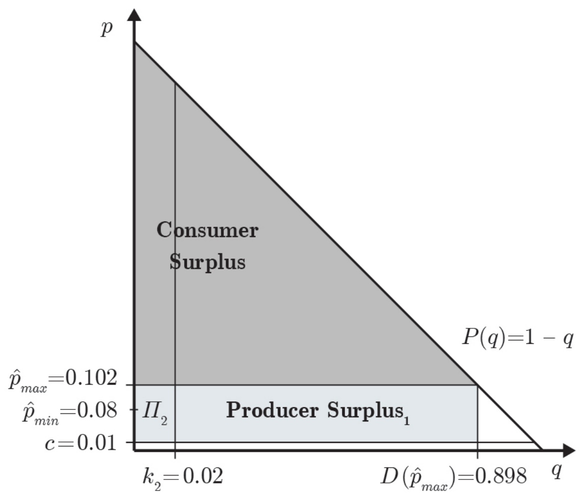

Figure 3: The weak private firm case - both firms have positive output

Clearly, for the equilibrium family $\pi_2(\cdot)$ is increasing in $\hat{p}$, while $\pi_1(\cdot)$ is decreasing in $\hat{p}$. The payoff intervals can also be calculated, in particular, $\pi_1 \in [0.4858, 0.4876]$, $\pi_2 \in [0.0014, 0.0018]$.

[15] In particular, if the private firm sets a price not greater than p_1^*, we are not anymore in Case C; if $q_2^* > D_2^r(p_2^*, q_1^*)$, then the private firm produces a superflous amount; if $q_2^* < D_2^r(p_2^*, q_1^*)$, then the private firm could still sell more than q_2^*; and if $q_2^* = D_2^r(p_2^*, q_1^*)$, then the private firm will choose a price-quantity pair maximizing profits with respect to its residual demand curve $D_2^r(\cdot, q_1^*)$.

[16] Because $p_2^m(\cdot)$ is a decreasing function in q_1

4.2 Public firm moves first

The case of public leadership is somewhat simpler. Namely, the firms clear the market in the only equilibrium family.[17] The results of public leadership are collected in the following proposition.

Proposition 5 (Public leadership)**.** *Assume that Assumptions 1-3, $q_2^m(k_1) \geq k_2$ and $P(k_1) > c$ hold. Then the prices and quantities associated with the pure strategy SPNE are*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, k_1, p^c, k_2)$$

where $p^* \in [0, P(k_1)]$.

Proof. We determine the reaction function $BR_2 = (p_2^*(\cdot,\cdot), q_2^*(\cdot,\cdot))$ of the private firm. Like in the strong private firm case, the private firm's best response correspondence can be obtained from the proof of Proposition 4, the corresponding simultaneous case.

$$BR_2(p_1, q_1) = \begin{cases} (p_1, \min\{k_2, D_2^r(p_1, q_1)\}) & \text{if } p_2^m(q_1) \leq p_1; \\ (p_2^m(q_1), q_2^m(q_1)) & \text{if } p_2^m(q_1) > p_1. \end{cases} \tag{17}$$

The reaction function dictates that the public firm maximizes social surplus in the first period by choosing any price level $p_1^* \leq p^c$ and quantity k_1. □

Recall the calculations of illuminating example of linear demand for the simultaneous-move case matching the actions associated with the only Nash-equilibrium in case of public leadership. Let the capacities and the unit cost be $k_1 = 0.9$, $k_2 = 0.02$ and $c = 0.01$. Then $p^c = 0.08$. The public firm will sell its entire capacity at a $p_1^* \in [0, p^c]$ market clearing price. The private firm will react with the market clearing price, and will also sell its entire capacity. This ensures the highest possible social surplus in this setting. Thus, for all $p_1 \in [0, 0.08]$ the actions associated with the SPNE are

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0.9, 0.08, 0.02),$$

where the corresponding payoffs are $\pi_1 = 0.4876$ and $\pi_2 = 0.0014$.

4.3 Private firm moves first

Finally, we consider the case of private leadership. The only pure-strategy equilibrium family of this case also appears in the simultaneous-moves subcase of the weak private firm case. Namely, the private firm produces on the original demand curve at the highest possible price level for which it is still in the public firm's interest to allow the private firm to do so. The equilibrium family is given formally in the following proposition.

Proposition 6 (Private leadership)**.** *Assume that Assumptions 1-3, $q_2^m(k_1) \geq k_2$ and $P(k_1) > c$ hold. Then the prices and quantities associated with the pure strategy SPNE are*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\}), \hat{p}, \min\{k_2, D(\hat{p})\})$$

where $p^* \in [0, \hat{p}]$, if and only if $\hat{p} \geq p^c$ and $p_2^* = \hat{p}$ is the highest price level for which

$$\pi_1(p^c, k_1, \hat{p}, \min\{k_2, D(\hat{p})\}) \leq \pi_1(p^*, D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\}), \hat{p}, \min\{k_2, D(\hat{p})\}) \tag{18}$$

[17] We speak about family, because p_1^* can vary within a given range

Proof. We determine the reaction function $BR_1 = (p_1^*(\cdot,\cdot), q_1^*(\cdot,\cdot))$ of the public firm. The public firm's best response correspondence can also be obtained from the proof of Proposition 4, the corresponding simultaneous-move case.

$$BR_1(p_2, q_2) = \begin{cases} (p^*, k_1) & \text{if (18) does not hold;} \\ (p_2, D_1^r(p_2, q_2)) & \text{if (18) holds.} \end{cases} \tag{19}$$

where $p^* \in [0, \hat{p}]$.

The reaction function dictates that the private firm maximizes its profit in the first period by choosing the highest possible price level, where the public firm is still better off (i.e. the social surplus is higher) by reacting with the same price and serving residual demand, than by undercutting p_2.[18] A highest price level $\hat{p}$ exists for every demand function, because if both firms choose price level p^c and sell their entire capacities (i.e. they clear the market), then Condition (18) always holds. □

Recall the calculations from the simultaneous-move case for linear demand. Let the capacities and the unit cost be $k_1 = 0.9$, $k_2 = 0.02$ and $c = 0.01$. Then $\tilde{p_2} = 0.102$. The private firm will choose $p_2^* = \tilde{p_2}$ and sells its entire capacity. The public firm will serve residual demand as it is not worth to undercutting the private firm's price which would cause superfluous production. Thus, for all $p_1 \in [0, 0.102]$ the actions associated with the only SPNE are

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0.878, 0.102, 0.02),$$

where the corresponding payoffs are $\pi_1 = 0.4858$ and $\pi_2 = 0.0018$.

5 The high unit cost case

The main assumption of this case is $c \geq P(k_1)$. In this case if the public firm produces at its capacity level, then the private firm will not enter the market because of the high cost level.

5.1 Simultaneous moves

In this subcase we have two types of pure-strategy Nash equilibria. The first type consists of profiles in which the private firm sets a price and produces a quantity on the residual demand curve, where in the particular case when the public firm does not produce anything in equilibrium, the residual demand curve coincides with the demand curve. In the second type, the public firm produces its capacity limit, while the private firm does not enter the market.

Proposition 7 (Simultaneous moves)**.** *Assume that $c \geq P(k_1)$ and Assumptions 1-3 hold. A strategy profile NE_1*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1^*, q_1^*, p_2^m(q_1^*), q_2^m(q_1^*))$$

is for any price-quantity pair

$$(p_1^*, q_1^*) \in \left\{(p_1, q_1) \mid 0 < q_1 < D(c), 0 \leq p_1 \leq p_2^d(q_1)\right\} \bigcup \tag{20}$$

$$\{(p_1, q_1) \mid q_1 = 0, 0 \leq p_1 \leq b\} \tag{21}$$

[18]Depending on the parameters, it can also occur that the public firm has zero output on the residual demand curve.

a Nash-equilibrium in pure strategies[19] *if and only if*

$$\pi_1(0, D(c), p_2^m(q_1^*), q_2^m(q_1^*)) \leq \pi_1(p_1^*, q_1^*, p_2^m(q_1^*), q_2^m(q_1^*)). \tag{22}$$

A strategy profile NE_2

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1^*, D(c), p_2^*, 0)$$

where $p_1^* \in [0, c]$*, and* $p_2^* \in [0, b]$*, also defines a Nash equilibrium family. Finally, no other equilibrium exists in pure strategies.*

Proof. Assume that $(p_1^*, q_1^*, p_2^*, q_2^*)$ is an arbitrary equilibrium profile. We divide our analysis into two subcases. In the first case the private firm is inactive (i.e. $q_2^* = 0$), while in the second case it is active on the market (i.e. $q_2^* > 0$)

Case A: Assume that $q_2^* = 0$, which means that only the public firm's production is positive, and since $c > P(k_1)$ it sets a price $p_1^* \leq c$ and quantity $q_1^* = D(c)$ in order to maximize social surplus. Therefore, only NE_2 type equilibria can emerge. We verify that indeed NE_2 specifies equilibrium profiles. Clearly, the public firm would reduce social surplus by switching unilaterally from its NE_2 strategy to a non NE_2 one. The private firm makes losses when producing a positive amount at a price $p_2^* < c$. In addition, $D_2^r(p_2^*, D(c)) = 0$ for all prices $p_2^* \geq c$ by $c > P(k_1)$ if the public firm plays an NE_2 strategy, and thus once again the private firm will just make losses if it produces a positive amount at a price $p_2^* \geq c$.

Case B: Assume that $q_2^* > 0$, which implies $p_2^* \geq c$ since otherwise the private firm would make losses. We divide our analysis into four subcases.

Subcase (i): Assume that $p_1^* = p_2^* > c$. Clearly, we cannot have $q_1^* + q_2^* < D(p_1^*)$ since otherwise the public firm could increase social surplus by increasing its production because of $c > P(k_1)$. Obviously, we cannot have $q_1^* + q_2^* > D(p_1^*)$ since then the public firm would have an incentive to reduce its production if $q_1^* > 0$ or the private firm could gain from decreasing its production if $q_1^* = 0$. In case of $q_1^* + q_2^* = D(p_1^*)$ we must have

$$q_2^* = \min\{k_2, D(p_2^*)\} \text{ and } q_1^* = D_1^r(p_2^*, \min\{k_2, D(p_2^*)\}) \tag{23}$$

since otherwise the private firm could radically increase its sales by a unilateral and sufficiently small price decrease.

Now we investigate when a strategy profile with prices $p_1^* = p_2^* > c$ and quantities given by (23) constitutes a Nash equilibrium profile. The private firm can benefit from setting higher prices if and only if $p_2^* < p_2^m(q_1^*)$. Moreover, the private firm can benefit from setting lower prices if and only if $p_2^* > p_2^m(q_1^*)$, which in fact can only be the case[20] if $q_1^* = 0$, because the private firm is not constrained by the production of the public firm by (23). Therefore, in a Subcase (i) equilibrium profile we must have $p_1^* = p_2^* = p_2^m(q_1^*) = p_2^d(q_1^*) > c$.[21] Clearly, if $q_1^* > 0$, then the public firm would decrease social surplus by a price increase (independently of a simultaneous quantity adjustment). If $q_1^* = 0$, then the public firm still will not benefit from setting higher prices. In addition, the public firm would not gain from setting a lower price if and only if (22) is satisfied.

To summarize, Subcase (i) admits those price-quantity pairs (p_1^*, q_1^*) from the set specified by (20) for which $p_1^* = p_2^d(q_1^*)$ results in equal prices.

[19] Recall that $q_1 < D(c) \Leftrightarrow P(q_1) > c$. In addition, $q_1 > 0$ implies $c < p_2^d(q_1) < p_2^m(q_1)$.

[20] If $k_2 \leq D(p_2^*)$, a price decrease cannot increase the private firm's profit, and if $k_2 > D(p_2^*)$, $q_1^* = 0$.

[21] Observe that this also implies $P(q_1^*) > c$.

Subcase (ii): Assume that $p_1^* = p_2^* = c$. As shown in Subcase (i) we must have $q_1^* + q_2^* = D(p_1^*)$. In addition, it can be easily checked that the private firm can benefit from a unilateral deviation if and only if $p_2^m(q_1^*) \in (c, a)$. Since $q_1^* < D(c)$ implies $p_2^m(q_1^*) \in (c, a)$ it follows that $q_1^* = D(c)$ should be the case, which would imply $q_2^* = 0$, leading to a departure from Case B. Hence, a Subcase (ii) equilibrium does not exist.

Subcase (iii): Assume that $p_1^* > p_2^* \geq c$. Then there cannot be an equilibrium in which $q_1^* > 0$ because the public firm could increase social surplus by switching to price p_2^* and quantity $(D(p_2^*) - q_2^*)^+$. Furthermore, in case of $q_1^* = 0$ we must have $q_2^* = D(p_2^*) \leq k_2$ since otherwise the public firm could again increase social surplus by switching to price p_2^* and quantity $(D(p_2^*) - q_2^*)^+$. Therefore, in a Subcase (iii) type equilibrium the private firm behaves as a monopolist, and thus $p_2^* = p_2^m(0)$ must be the case, which in turn is an equilibrium if and only if the public firm has no incentive to enter the market, that is (22) is satisfied.

Observe that the derived equilibrium is an NE_1 type equilibrium and the respective price-quantity pairs (p_1^*, q_1^*) are a subset of the set specified by (21).

Subcase (iv): Assume that $p_1^* < p_2^*$ and $p_2^* \geq c$. In case of $D_2^r(c, q_1^*) = 0$ we must have $q_2^* = 0$, which has been already investigated in Case A. Therefore, in what follows we can assume that $D_2^r(c, q_1^*) > 0$, which in turn implies that $p_2^m(q_1^*) \in (c, a)$ and that $p_2^d(q_1^*) \in (c, p_2^m(q_1^*))$ is well defined. Observe that we must have $q_1^* + q_2^* = D(p_2^*)$ since otherwise, for instance, the public firm could increase social surplus by either increasing or decreasing its output. It can be checked that the private firm does not undercut the public firm's price if and only if $p_1^* \leq p_2^d(q_1^*)$. Moreover, if the private firm does not undercut the public firm's price, then it will set price $p_2^m(q_1^*)$ and quantity $q_2^m(q_1^*)$. The derived strategy profile constitutes a Nash equilibrium profile if and only if the public firm has no incentive to deviate, that is (22) is satisfied.

It can be checked that we have determined an NE_1 type equilibrium and the respective price-quantity pairs (p_1^*, q_1^*) lie in the set specified by (20), where $q_1^* > 0$ and $p_1^* \in [0, p_2^d(0)] \subset [0, p_2^m(0)]$ resulting in a higher price for the private firm.[22] □

Pick the capacities and unit cost levels $k_1 = 0.5$, $k_2 = 0.1$, $c = 0.6$ and let $D(p) = 1 - p$, which lead to the high unit cost case. We give examples to the equlibria in the order they are listed in the statement. Firstly, from these exogenously given values we can calculate the interval where $\hat{p}$ can be taken from, leading to Nash equilibria which are not payoff equivalent: $\hat{p} \in [0.6, 0.8]$. We can choose $\hat{p} = 0.8$ (see Figure 4). This leads to the following values: $p_1^* \in [0, 0.8]$; $q_1^* = 0.1$; $p_2^* = 0.8$; $q_2^* = 0.1$. In this case $\pi_1 = 0.06$ (sum of dark and light gray areas); $\pi_2 = 0.04$ (light gray area indicated by π_2).

Depending on $\hat{p}$, profit levels can vary in the following intervals: $\pi_1 \in [0.06, 0.08]$ and $\pi_2 \in [0, 0.04]$.

Turning to the second equilibrium type, where the private firm is not present on the market, we obtain $p_1^* \in [0, 0.5]$; $q_1^* = 0.5$; $p_2^* \in \mathbb{R}$; $q_2^* = 0$. Profit levels are as follows: $\pi_1 = 0.08$; $\pi_2 = 0$.

Finally, for the illustration of the third equilibrium we have that any $q_1 \in [0, k_1]$ leads to a Nash equilibrium. Let us fix $q_1 = 0.3$. Now $p_2^m(0.3) = 0.65$ an $p_2^d(0.3) = 0.325$. Thus, $p_1^* \in [0, 0.325]$; $q_1^* = 0.3$; $p_2^* = 0.65$; $q_2^* = 0.05$. In this case, $\pi_1 = 0.0787$ and $\pi_2 = 0.0013$.

[22] It can be verified that we have obtained all NE_1 type equilibria.

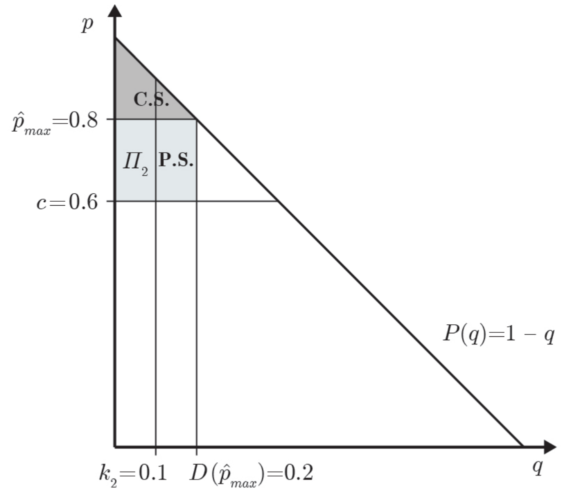

Figure 4: The high unit cost case - both firms have positive output - case 1

Depending on q_1, profit levels can vary in the following intervals: $\pi_1 \in [0.06, 0.08]$ and $\pi_2 \in [0, 0.04]$.[23]

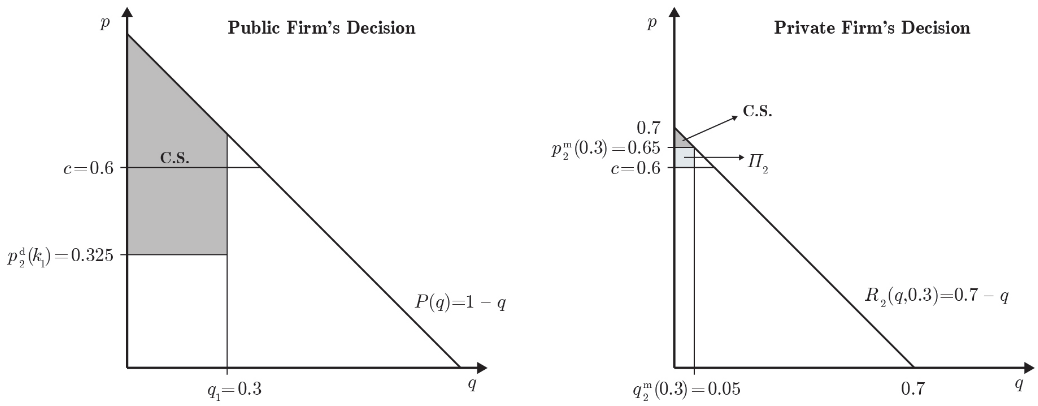

Figure 5: The high unit cost case - both firms have positive output - case 2

5.2 Public firm moves first

In the high unit cost case with public leadership we obtain that the private firm does not enter the market, while the public firm's output equals its capacity. This result is formalized in the following proposition.

[23]We note that here $p_1^* < c$, still, it is of the public firms interest to produce a positive amount, as this action leads to a positive change in consumer surplus. This is the reason why there is no producer surplus indicated on the left-hand-side of Figure 5.

Proposition 8 (Public leadership)**.** *Assume that $c > P(k_1)$ and Assumptions 1-3 hold. Then the prices and quantities associated with the pure strategy SPNE are*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, D(c), p_2^*, 0)$$

where $p^* \in [0, c]$ and $p_2^* \in [0, b]$.

Proof. We determine the reaction function $BR_2 = (p_2^*(\cdot,\cdot), q_2^*(\cdot,\cdot))$ of the private firm. The private firm's best response correspondence can be obtained from the proof of Proposition 7, the corresponding simultaneous-move case.

$$BR_2(p_1, q_1) = \begin{cases} \{(p, 0) \mid p \in [0, b]\} & \text{if } D(c) \leq q_1 \leq k_1 \text{ and } p_1 \leq c; \\ \{(p_1, \min\{k_2, D(p_1)\})\} & \text{if } p_2^d(q_1) < p_1 \text{ and } p_1 > c; \\ \{(p_1, \min\{k_2, D(p_1)\})\} \cup & \\ \quad \{(p_2^m(q_1), q_2^m(q_1))\} & \text{if } p_2^d(q_1) = p_1 \text{ and } p_1 > c; \\ \{(p_2^m(q_1), q_2^m(q_1))\} & \text{if } p_2^d(q_1) \geq p_1 \text{ and } p_1 > c. \end{cases} \tag{24}$$

Note that the above four areas partition $[0, b] \times [0, k_1]$ since $q_1 < D(c)$ implies $p_2^d(q_1) > c$. From the derived reaction function it follows that the public firm maximizes social surplus in the first period by choosing any price level $p^* \in [0, c]$ and quantity k_1. □

Recall the outcome of the simultaneous case when setting the parameters to $k_1 = 0.5$, $k_2 = 0.1$, and $c = 0.6$, and picking demand curve $D(p) = 1 - p$. Then the the private firm is not present on the market, and we obtain $p_1^* \in [0, 0.5]$, $q_1^* = 0.5$, $p_2^* \in \mathbb{R}$, and $q_2^* = 0$. Payoffs equal $\pi_1 = 0.08$ and $\pi_2 = 0$.

5.3 Private firm moves first

Finally, we consider the case of private leadership. We will establish for this case that in equilibrium the private firm chooses the highest price level at which the public firm does not capture the entire market at price c or smaller. The respective price is determined either as the price at which the public firm is indifferent between matching the private firm's price and capturing the entire market at a price less than or equal to c and producing $D(c)$, despite the fact that the production of the private firm may be wasted, or by the private firm's monopoly price.

Proposition 9 (Private leadership)**.** *Assume that $c \geq P(k_1)$ and Assumptions 1-3 hold. Then there exists a unique price $p_2^* \in (c, p_2^m(0)]$ such that the prices and quantities associated with the pure strategy SPNE are*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1^*, D_1^r(p_2^*, \min\{k_2, D(p_2^*)\}), p_2^*, \min\{k_2, D(p^*)\})$$

where $p_1^* \in [0, p_2^*]$,

Proof. Clearly, if the private firm does not produce anything, i.e. $q_2 = 0$, then the public firm follows with $(p_1, D(c))$ such that $p_1 \leq c$. If the private firm's production is positive, i.e. $q_2 > 0$, then we must have $p_2 \geq c$. Furthermore, the private firm never produces more than $D(p_2)$.

Focusing on the SPNE, we determine the best replies of the public firm only to the first-stage actions of the private firm lying in

$$A = \{(p_2, 0) \mid p_2 \in [0, b]\} \cup \{(p_2, q_2) \mid p_2 \in [c, b] \text{ and } q_2 \in (0, D(p_2)]\}.$$

For a given $(p_2, q_2) \in A$ such that $q_2 \in (0, D(p_2)]$ the public firm never sets a price above p_2 if it decides to produce at all, i.e. $q_1 > 0$. Moreover, in the latter case the public firm's production has to equal $q_1 = D_1^r(p_2, q_2)$, since if it does not capture the entire market, social surplus will be determined at price p_2 and superfluous production decreases social surplus. Therefore, the response of the public firm is determined by inequality

$$\pi_1(0, D(c), p_2, q_2) \leq \pi_1(c, D_1^r(p_2, q_2), p_2, q_2), \tag{25}$$

where its response equals $BR_1(p_2, q_2) = \{(p_1, D_1^r(p_2, q_2)) \mid p_1 \leq c\}$ if $q_2 > 0$ and (25) is satisfied, and $BR_1(p_2, q_2) = \{(p_1, D(c) \mid p_1 \leq c\}$ if $q_2 = 0$ and (25) is violated.[24]

Taking the best responses of the public firm into consideration, the private firm will produce $q_2 = \min\{k_2, D(p_2)\}$ at price p_2 if (25) is satisfied.[25] By substituting $q_2 = \min\{k_2, D(p_2)\}$ into (25) it follows that the right-hand side of (25) is continuous, strictly decreasing in p_2 on $[c, p_2^m(0)]$, and it is larger than its left-hand side at price $p_2 = c$. Since the private firm does not set a price above $p_2^m(0)$ it will either set the price in $(c, p_2^m(0))$ for which (25) is satisfied with equality or price $p_2^m(0)$. □

Pick linear demand and let the capacities and the unit costs be $k_1 = 0.5$, $k_2 = 0.1$, $c = 0.6$. Then as it can be determined $\hat{p} = 0.8$. This leads us to $p_1^* \in [0, 0.8]$; $q_1^* = 0.1$; $p_2^* = 0.8$; $q_2^* = 0.1$, which implies $\pi_1 = 0.06$; $\pi_2 = 0.04$.

6 Solution of the timing game

We consider a timing in which the firms in stage 1 can choose between two periods for the announcement of their price and quantity decision. Thereafter, knowing each others timing decision, the firms in stage 2 set their prices and quantities in the selected periods.

The equilibrium of the timing game can be derived from Propositions 1-9, by comparing the payoffs of both firms for different orderings of moves.

Before we turn to the solution of the timing game, we provide a summary of the payoffs that were calculated in the numerical examples after Propositions 1-9, respectively. Table 1 provides numerical evidence of the solution of the timing game for the particular demand function $D(p) = 1 - p$, with exogenously given capacities and cost levels.

It is easy to see from Table 1 that in all the three main cases any firm has the highest payoff with certainty in case it is the first mover. Thus, as every firm wants to become the leader and there cannot be two leaders at the same time, the outcome of the timing game is simultaneous moves. The equilibrium of the timing game for any concave, twice continuously differentiable demand function is precisely stated in the following proposition.

Proposition 10. *Assume that Assumptions 1-3 hold. For any cost and capacity levels, the equilibrium of the timing game lies at simultaneous moves.*

Proof. The result comes directly from Propositions 1-9. □

7 Corollaries and concluding remarks

Our main results are collected in the following corollaries. We focus on the differences between the production-to-order case - which was investigated in earlier work - and the

[24] To be precise if (25) is satisfied with equality, then both mentioned types are best responses; however, as it can be verified in a SPNE only the former type can be selected.

[25] Note that the distribution of production between the two firms does not effect (25).

Cases	**Strong private firm**	**Weak private firm**	**High unit cost**
$\mathbf{k_1}$	0.5	0.9	0.5
$\mathbf{k_2}$	0.4	0.02	0.1
$\mathbf{c}$	0.1	0.01	0.6
π_1: **Public firm's equilibrium payoff (social surplus)**			
sim. moves	$\in [0.28, 0.435]$	$\in [0.4858, 0.4876]$	$\in [0.06, 0.08]$
as leader	0.435	0.4876	0.08
as follower	0.28	0.4858	0.06
π_2: **Private firm's equilibrium payoff (profit)**			
sim. moves	$\in [0.04, 0.2]$	$\in [0.0014, 0.0018]$	$\in [0, 0.04]$
as leader	0.2	0.0018	0.04
as follower	0.04	0.0014	0

Table 1: Example payoff levels for the demand function $D(p) = 1 - p$

production-in-advance case from the point of view of equilibrium strategies, social surplus effects and equilibrium analysis of the timing game. The first corollary determines the endogenous order of moves in a two-period timing game of the production-in-advance framework, where both firms can choose between two periods for setting their prices and quantities.

Corollary 1. *In the production-in-advance framework both firms want to become the first mover, therefore the equilibrium of the timing game lies at simultaneous moves.*

We turn to the problem of the public firm's influence on social surplus. One can carry out a comparison with the results for the production-to-order case presented in Balogh and Tasnádi (2012). In the PIA case the social surplus becomes lower - let them play any pure-strategy Nash equilbria - than that of the PTO case. This result is put down in the next corollary.

Corollary 2. *When playing the production-in-advance type of the Bertrand-Edgeworth game, the equilibrium strategies lead to a decrease in social surplus compared to the PTO case.*

The third main result of the paper is implicitely given in Section 5: independently from the parameters and the orderings of firms' decisions, the production-in-advance type Bertrand-Edgeworth mixed duopoly always has at least one pure-strategy Nash equilibrium. This result remained the same as that of the mixed PTO case. However, we emphasize that in case of standard Bertrand-Edgeworth duopolies, there is a lack of pure-strategy equilibria (see e.g. Deneckere and Kovenock (1992)). We state the existence of a pure-strategy equilibrium in the third corollary.

Corollary 3. *We have at least one pure-strategy (subgame-perfect) Nash equilibrium in all three analyzed cases and for all three orderings of moves.*

These results are summarized in the following table.

The results suggest that it is by far not all the same whether a public firm has some influence on an oligopoly market. Further research directions may include the application of our model to markets with asymmetric information, partial public ownership, and

	Production-to-order	*Production-in-advance*
Equilibrium in pure strategies	Yes	Yes
Timing game equlibrium	All possible orderings	Simultaneous moves
Public firms's social surplus effect	Positive	Negative[26]

Table 2: Comparison of the PTO and PIA cases

oligopolies with more than two firms. One can notice that our assumptions were quite general in the present paper. However, to present plausible results in the mentioned topics, more strict assumptions may be needed.

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